A Lattice on Dyck Paths Close to the Tamari Lattice

17th workshop on Computational Logic and Applications
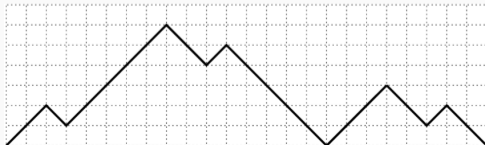arxiv.org/abs/2309.00426

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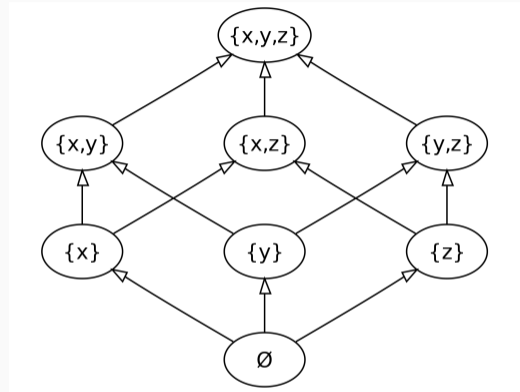
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Introduction

A partial order, is a binary relation $\leq$ on a set P such that for all $a, b, c \in P$

- $a \leq a$ (reflexivity)
- $a \leq b$ and $b \leq a \implies a = b$ (antisymmetry)
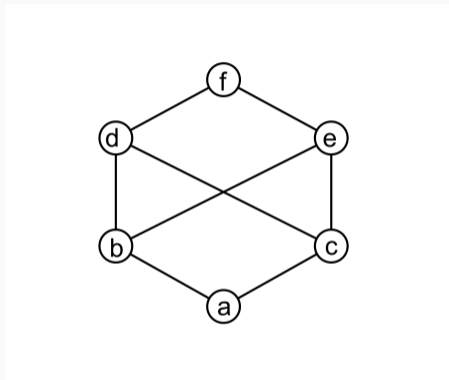- $a \leq b$ and $b \leq c \implies a \leq c$ (transitivity)

set of all subsets of 3 elements ordered by inclusion

Let $(P, \leq)$ be a partially ordered set. Let $x, y, m \in P$, then m is called the **Meet** (greatest lower bound or infimum) $m = x \wedge y$ if :

- $m \leq x$ and $m \leq y$
- For any $w \in P$, with $w \leq x$ and $w \leq y$ then $w \leq m$
- Dually a **Join** (Smallest upper bound or supremum) $m = x \vee y$
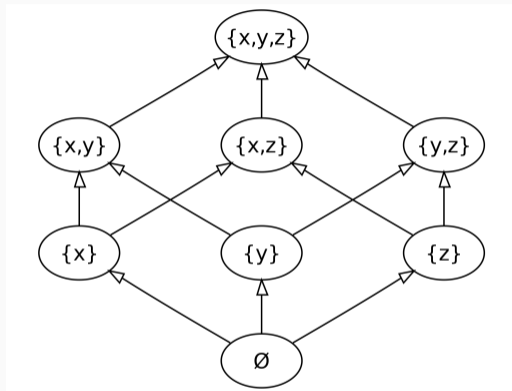- If a **meet** (resp. **join**) exists then it is **unique**

Meets do not always exist (for example d, e)

A partially ordered set $(L, \leq)$, is a lattice if $\forall a, b \in L$
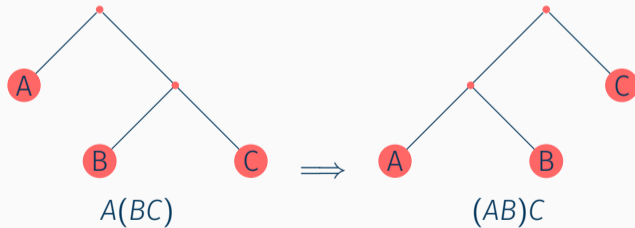
- a, b have a infimum ($a \wedge b$ exists)
- a, b have an supremum ($a \vee b$ exists)

set of all subsets is a lattice

- The Tamari Lattice is a poset introduced by Dov Tamari in 1962
- The Poset has equivalent definitions on bracketed expressions, binary trees, Dyck paths and triangulations
- Many connections with triangulations, combinatorial maps, lambda calculus, …

$$A(BC) \implies (AB)C$$

If we denote by $\mathcal{T}_n$ the set of bracketed expressions with n atoms.

Definition

The Tamari poset by endowing $\mathcal{T}_n$ with the transitive closure $\preceq$ of the covering relation $A(BC) \longrightarrow (AB)C$ (shifting a parenthesis to the left)
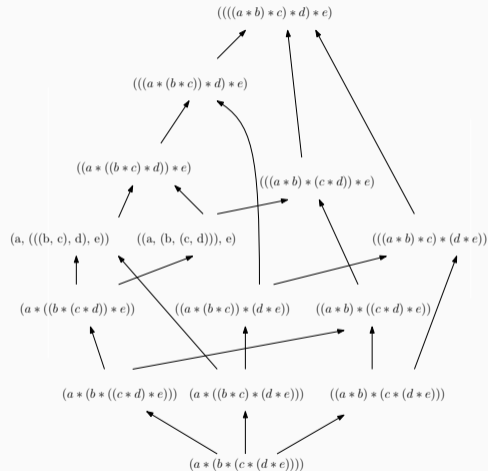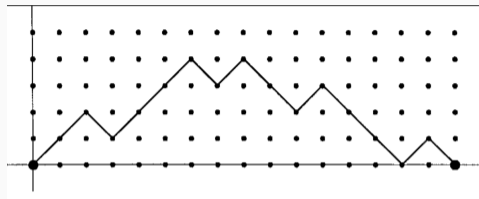
A Dyck path is a lattice path in $\mathbb{N}^2$ starting at the origin, ending on the x-axis and consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$.

Catalan numbers

Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, then :

$$|\mathcal{D}_n| = (2n)!/(n!(n+1)!)$$

$\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$

- **first/last return decomposition** of a non-empty Dyck path is unique, $P = URDS$, where $R, S \in \mathcal{D}$

- A Dyck path is **prime** whenever it only touches the x-axis at its beginning and its end

Tamari Lattice

Defined by endowing $\mathcal{D}_n$ with the transitive closure $\preceq$ of the covering relation transforming an occurrence of *DUQD* into an occurrence *UQDD* where $Q \in \mathcal{D}$.
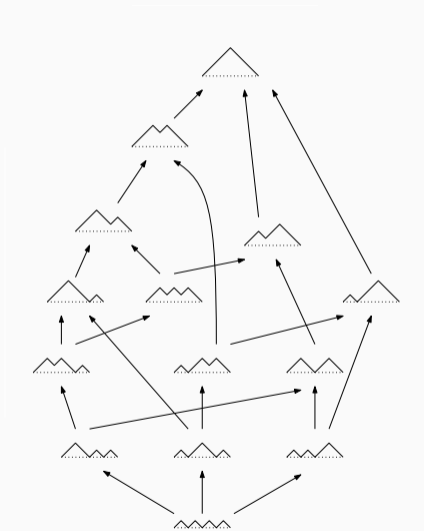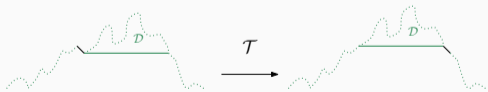
BKN Poset

BKN poset

Defined by endowing $\mathcal{D}_n$ with the transitive closure $\leq$ of the covering relation transforming an occurrence of $DU^k D^k$ into an occurrence $U^k D^k D$ with $k \geq 1$.
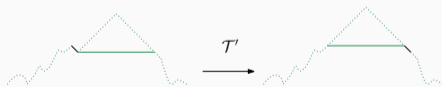
The red arrow does not belong to BKN
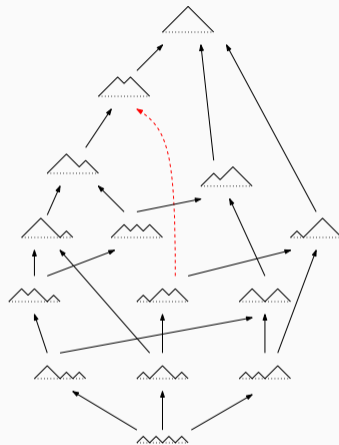
Propositions :

1. The poset $(\mathcal{D}_n, \leq)$ admits a maximum element and a minimum element.

2. Given $P, Q \in \mathcal{D}_n$ satisfying $P \leq Q$, $P \neq Q$, and such that $P = RDS$ and $Q = RUS'$ (R is the maximal common prefix). Let W the Dyck path obtained from P by applying the covering $P \longrightarrow W$ on the leftmost occurrence of $DU^k D^k$, $k \geq 1$, in DS, then we necessarily have $W \leq Q$.

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

For $n \leq 3$ the poset is isomorphic to the Tamari lattice.

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

Assume $\mathcal{S}_n = (\mathcal{D}_n, \leq)$ is a lattice for $n \leq N$, and show for $N + 1$. Distinguish according to first return decomposition

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

(1) If $P = URDS$ and $Q = UR'DS'$ where $|R| = |R'|$. Apply the recurrence hypothesis for R and R' (resp. S and S'), which means that $R \vee R'$ (resp., $S \vee S'$) exists. Then, the path $U(R \vee R')D(S \vee S')$ is necessarily the least upper bound of P and Q, proving existence of $P \vee Q$.

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

(2) Let us suppose that $P = URDS$ and $Q = UR'DS'$ where $|R'| < |R|$. Let M be an upper bound of P and Q (Prop 1). Since $|R'| < |R|$, M has necessarily a decomposition $M = UM_1DM_2$ where $|M_1| \geq |R|$. In any sequence of coverings $Q \to \ldots \to M$
from Q to M, there is necessarily a covering that elevates the down-step just after R'

$$Q = \quad \leq \quad \longrightarrow \quad = Q_1$$

Theorem

The poset $(\mathcal{D}_n, \leq)$ is a lattice

Existence of a join element. By induction on the semilength of the Dyck paths.

Iterating this process with P and Q_1, construct P', Q' such that $P \leq M, Q \leq M$ $\equiv P' \leq M, Q' \leq M$ where P' and Q' lie **(1)**. Using the hypothesis recurrence $P' \vee Q' = P \vee Q$ exists. The existence of greatest lower bound then follows automatically since the poset is finite with a least and greatest elements. □

Characteristics of BKN

Let $A(x, y, z) = \sum_{n \geq 0} a_{n,k,\ell} x^n y^k z^\ell$ be the generating function where $a_{n,k,\ell}$ is the number of Dyck paths of

- semilength n having
- k possible coverings (or equivalently k outgoing edges),
- ℓ incoming edges.

$$A(x, y, z) = \frac{R(x, y, z) - \sqrt{4x(xzy - xy - xz + 1)(xy + xz - x - 1) + R(x, y, z)^2}}{2x(xzy - xy - xz + 1)},$$

where $R(x, y, z) = x^2 zy - x^2 y - x^2 z + x^2 - xy - xz + x + 1$.

$A(x, y, z) = \frac{R(x,y,z) - \sqrt{4x(xzy - xy - xz + 1)(xy + xz - x - 1) + R(x,y,z)^2}}{2x(xzy - xy - xz + 1)}$, where

$R(x, y, z) = x^2zy - x^2y - x^2z + x^2 - xy - xz + x + 1$.

- Using last return decomposition $P = RUSD$
- 6 different cases according to R and S

$$A = 1 + \underbrace{x}_{\substack{R=S=\epsilon}} + \underbrace{(A-1)xy}_{\substack{R\neq\epsilon \\ S=\epsilon}} + \underbrace{\frac{x^2z}{1-xz}}_{\substack{R=\epsilon \\ S=U^\alpha D^\alpha}} + \underbrace{\frac{x^2z}{1-xz}(A-1)y}_{\substack{R\neq\epsilon, S=U^\alpha D^\alpha}} + \underbrace{\frac{x^2z}{1-xz}(A-1)yA}_{\substack{S=S'U^\alpha D^\alpha}}$$

$$+ Ax\underbrace{\left(A - 1 - x - \frac{x^2z}{1-xz} - x(A-1)y - \frac{x^2z}{1-xz}(A-1)y\right)}_{S\neq S'U^\alpha D^\alpha},$$

(1)

G.F $E(x)$ of the total number of possible coverings over all Dyck paths of semilength n (or equivalently the number of edges in the Hasse diagram) is

$$E(x) = \frac{-1 + 4x + (1 - 2x)\sqrt{1 - 4x}}{2(1 - 4x)(1 - x)}.$$

From $A(x, y, z)$ simply compute $\partial_y(A(x, y, 1))|_{y=1}$.
The coefficients of x^n (A057552 in [Sloane et al., 2003]) are given by

$$\sum_{k=0}^{n-2} \binom{2k + 2}{k}.$$

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The ratio between the numbers of coverings in $\mathcal{T}_n$ and $\mathcal{S}_n$ tends towards $3/2$.

- An interval is an ordered pair of elements (P, Q) with $P \leq Q$

- Inspired by [Bousquet-Mélou and Chapoton, 2023]

- Let $I(x, y) = \sum\limits_{n,k \geq 1} a_{n,k} x^n y^k$, where $a_{n,k}$ number of intervals in $\mathcal{S}_n$ with upper path ends with k down-steps exactly

- Let $J(x, y) = \sum\limits_{n,k \geq 1} b_{n,k} x^n y^k$, where $b_{n,k}$ number of intervals (P, Q) in $\mathcal{S}_n$ such that the upper path Q is **prime** and ends with k down-steps exactly
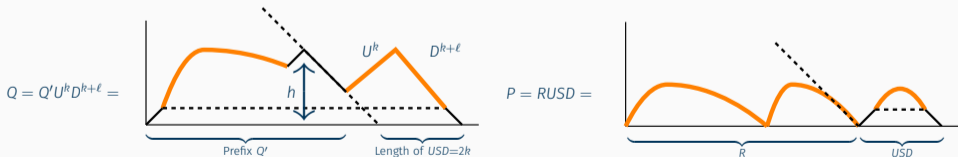
$$I(x, y) = \underbrace{J(x, y)}_{\substack{\text{Interval is} \\ \text{either prime}}} + \underbrace{I(x, 1) \cdot J(x, y)}_{\substack{Q = RUSD, P = P_1 P_2 \\ I_1 := (P_1, R) \text{ and } I_2 := (P_2, USD)}} \qquad (2)$$

The following also holds :

$$J(x, y) = \underbrace{xy}_{\substack{\text{P=UD and Q=UD}}} + \underbrace{xyI(x, y)}_{\substack{P \text{ is prime, } P = UP'D \\ \text{and necess. } Q = UQ'D}} + \underbrace{\frac{J(x, y) - J(x, 1)}{y - 1} \cdot C(xy)xy^2}_{\substack{P \text{ is not prime, } P = RUSD \\ \text{const. } h \text{ intervals}}}, \quad (3)$$

where $C(x)$ is the g.f. for Catalan numbers, i.e., $C(x) = 1 + xC(x)^2$.

With little rearrangments

$$\begin{cases} I(x,y) &= \frac{J(x,y)}{1-J(x,1)}, \\ J(x,y) &= xy + xy\frac{J(x,y)}{1-J(x,1)} + \frac{J(x,y)-J(x,1)}{y-1} \cdot C(xy)xy^2. \end{cases}$$

In order to compute $J(x,1)$ use the kernel method [Banderier et al., 2002] on

$$J(x,y) \cdot \left(1 - \frac{xy}{1-J(x,1)} - \frac{C(xy)xy^2}{y-1}\right) = xy - \frac{J(x,1)}{y-1} \cdot C(xy)xy^2.$$

Cancel the factor of $J(x,y)$ by finding y as a function y_0 of $J(x,1)$ and x to find :

$$\begin{cases} 1 - \frac{xy_0}{1-J(x,1)} - \frac{C(xy_0)xy_0^2}{y_0-1} &= 0, \\ xy_0 - \frac{J(x,1)}{y_0-1} \cdot C(xy_0)xy_0^2 &= 0. \end{cases}$$

Then $y_0 = \frac{1+4x-\sqrt{1-8x}}{8x}$.

- The generating function $J(x, y)$ can be found explicitly
- From $J(x, y)$ we exhibit
  (prime intervals) $J(x, 1) = \frac{1-\sqrt{1-8x}}{4} = x + 2x^2 + 8x^3 + 40x^4 + 224x^5 + \ldots$
  (A052701) $(2^{n-1}c_{n-1})$
- We then obtain : $I(x, y) = J(x, y) \cdot \frac{3-\sqrt{1-8x}}{2(x+1)}$
- **(intervals)** $I(x, 1) = \frac{1-2x-\sqrt{1-8x}}{2(x+1)} = x + 3x^2 + 13x^3 + 67x^4 + 381x^5 + \ldots$

  (A064062) $\left(\frac{1}{n} \sum\limits_{m=0}^{n-1} (n-m)\binom{n+m-1}{m}2^m\right)$

<div align="center">

Both sequences are related to counting outerplanar
maps [Geffner and Noy Serrano, 2017]

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Asymptotic exponential growth of intervals in $\mathcal{T}_n$ and $\mathcal{S}_n$ is $\left(\frac{32}{27}\right)^n$

Open questions

- In [Zeilberger, 2019] showed a sequent calculus capturing the Tamari order (semi-associative law)
- Can we find a calculus capturing the BKN order?
- Currently working on proofs

$$\overline{A \Rightarrow A} \; \text{id}$$

$$\frac{A, B, \Delta \Rightarrow C}{A * B, \Delta \Rightarrow C} \; L$$

$$\frac{\Delta \Rightarrow A \qquad \Gamma \Rightarrow B}{\Delta, \Gamma \Rightarrow A * B} \; R$$

- A, B, C are formulas, Δ, Γ are lists of formulas
- **(atoms)** lowercase latin letters
- **(Formulas)** $\mathcal{F} \coloneqq a, b, \ldots \mid (\mathcal{F} * \mathcal{F})$

$$\cfrac{\cfrac{}{b \Rightarrow b}\ \text{id} \quad \cfrac{}{c \Rightarrow c}\ \text{id}}{b, c \Rightarrow (b * c)}\ \text{R1} \qquad \cfrac{\cfrac{}{d \Rightarrow d}\ \text{id} \quad \cfrac{}{e \Rightarrow e}\ \text{id}}{d, e \Rightarrow (d * e)}\ \text{R1}$$

$$\cfrac{b, c, d, e \Rightarrow ((b * c) * (d * e))}{(b * c), d, e \Rightarrow ((b * c) * (d * e))}\ \text{L}\ \ \text{R2}$$

$$\cfrac{}{a \Rightarrow a}\ \text{id}$$

$$\cfrac{a, (b * c), d, e \Rightarrow (a * ((b * c) * (d * e)))}{(a * (b * c)), d, e \Rightarrow (a * ((b * c) * (d * e)))}\ \text{L}$$

$$\cfrac{((a * (b * c)) * d), e \Rightarrow (a * ((b * c) * (d * e)))}{(((a * (b * c)) * d) * e) \Rightarrow (a * ((b * c) * (d * e)))}\ \text{L}$$

$$\cfrac{}{A \Rightarrow A}\ \text{id}$$

$$\cfrac{A, B, \Delta \Rightarrow C}{A * B, \Delta \Rightarrow C}\ \text{L}$$

$$\cfrac{\Delta \Rightarrow A \quad C \Rightarrow B}{\Delta, C \Rightarrow A * B}\ \text{R1}$$

$$\cfrac{\mathfrak{T} \Rightarrow A \quad \Delta \Rightarrow B}{\mathfrak{T}, \Delta \Rightarrow A * B}\ \text{R2}$$

A, B, C are formulas, Δ a list of formulas and $\mathfrak{T}$ a list of atoms.

- In [Zeilberger, 2019] showed that **Tamari intervals** are in bijection with **Closed indecomposable β-normal linear planar lambda terms**

- BKN Lattice being a restriction of the Tamari Lattice

- Can we characterize the properties of the fragment of Lambda Calculus induced by BKN?

$\lambda e.(\lambda d.(\lambda c.(\lambda a.(((a)@(\lambda b.((b)@(c))))@((d)@(e))))))$
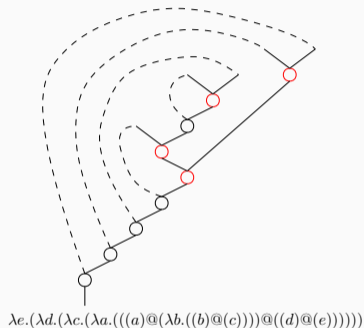
First term belonging to Tamari but not to BKN

- Sequence of prime intervals :
  $x + 2x^2 + 8x^3 + 40x^4 + 224x^5 + 1344x^6 + \ldots$ (A052701) $(2^{n-1}c_{n-1})$

- Also corresponds to Number of Dyck paths of semilength n in which the step $U = (1, 1)$ not on ground level comes in 2 colors

- Can we find a bijection between these classes?
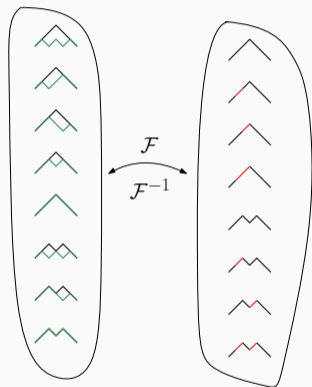
- The *diameter* is the maximum distance between any two vertices
- The diameter of BKN gives an upper bound on the diameter of the Tamari Lattice
- For $n \geq 3$, we conjecture that the diameter of $\mathcal{S}_n$ is $2n - 4$, and that this value corresponds to the distance between $(UD)^n$ and $UU(UD)^{n-2}DD$.
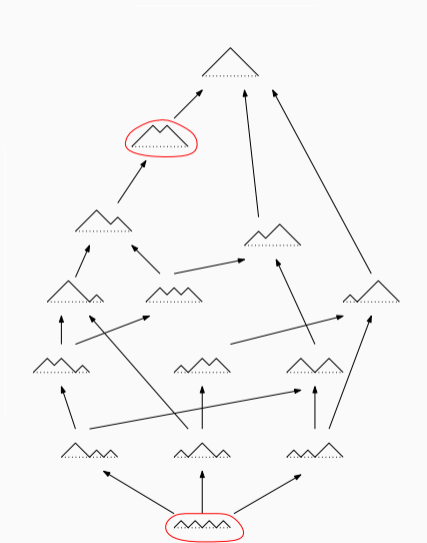
Banderier, C., Bousquet-Mélou, M., Denise, A., Flajolet, P., Gardy, D., and Gouyou-Beauchamps, D. (2002).
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A sequent calculus for a semi-associative law.
Logical Methods in Computer Science, 15.

$$J(x,y) = \frac{xy\,(-1+J(x,1))\,(J(x,1)\,C\,(xy)\,y-y+1)}{J(x,1)\,C\,(xy)\,xy^2 - C\,(xy)\,xy^2 - xy^2 - J(x,1)\,y + xy + J(x,1) + y - 1}$$