Introduction to abstract interpretation
Part I

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Introduction

Points du jour

- Welcome new young people in the lab.

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- Suggestion for next year.
- Upcoming seminars: part II of this talk and Dina's talk.

Material

- My slides are based mostly on Antoine Miné's lecture notes (freely available on his webpage) and
- two video courses link and link.

Introduction

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Image for illustration purposes only

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"Testing is not sufficient!"

- There are many famous documented bugs even in well tested softwares.
- We should use formal methods to provide rigorous, mathematical insurance of correctness even though we can not prove everything.
- Correctness properties are undecidable! An automatic general method is impossible to find (we have to find compromises).

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Formal methods

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Cousot and Cousot in [CC10] classify current formal methods into three categories

- Deductive methods.
- Model Checking.
- Static Analysis.

Abstract interpretation is a theory of approximation and analysis of program semantics that belongs to the category of *Static Analysis*.

Deductive Methods

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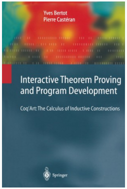
- Examples include proof assistants such as Coq developed by Bertot and Castéran in [BC13].



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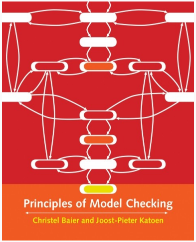

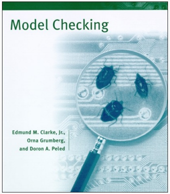

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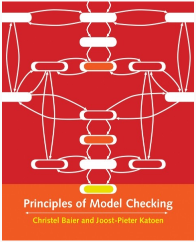

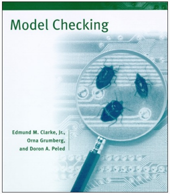

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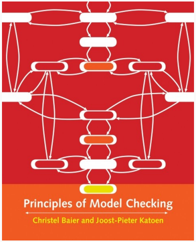

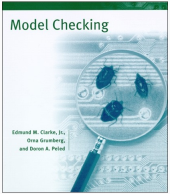

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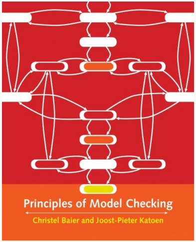

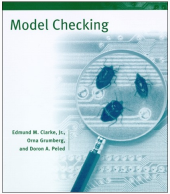

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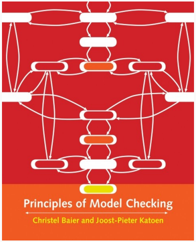
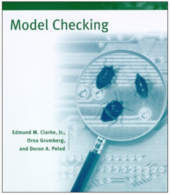
- Software Examples include CBMC.





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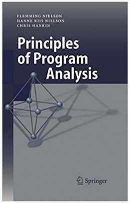

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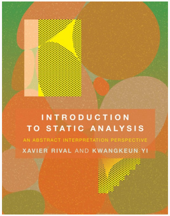

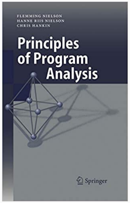

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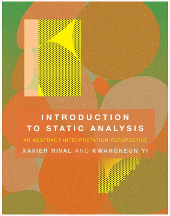

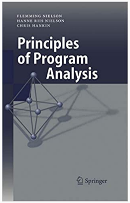

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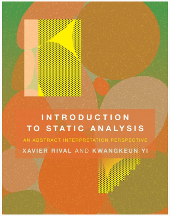

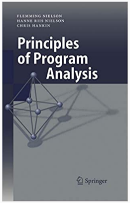

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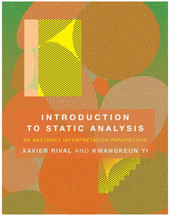
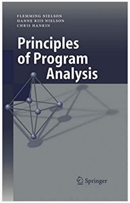
- Signals all possible Run Time Errors (soundness). This includes division by zero, bounds array indexing, integers and arithmetic overflow. . .
- It also signal some errors that cannot really happen (false alarms on spurious executions e.g. when hypotheses on the execution environment are not taken into account).





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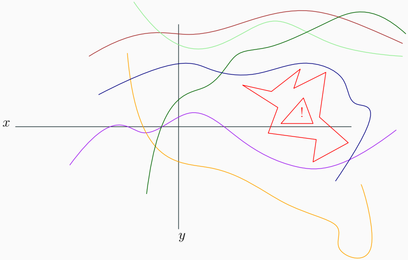

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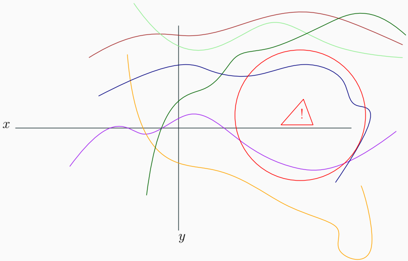

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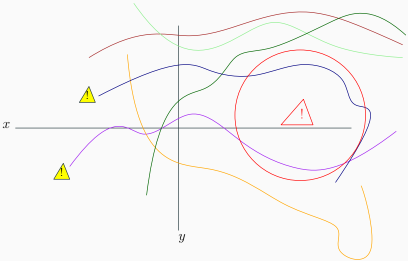

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Order Theory

Partial orders

Definition (Partial Order, Poset)

A partial order $\sqsubseteq$ on a set X is a binary relation over X such that:

- reflexive: $\forall x \in X, x \sqsubseteq x$;
- anti-symmetric: $\forall x, y \in X, (x \sqsubseteq y) \wedge (y \sqsubseteq x) \implies (x = y)$;
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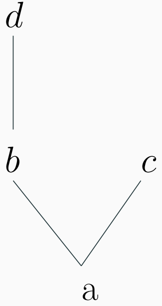
Let $X = \{a, b, c, d\}$, then $R = \{(a, a), (b, b), (c, c), (d, d)(a, b), (a, c), (b, d), (a, d)\}$ is a partial order.

Representation with Hasse Diagram.



Poset examples

$\dots$

3

2

1

0

Integer poset (totally ordered)

Poset examples

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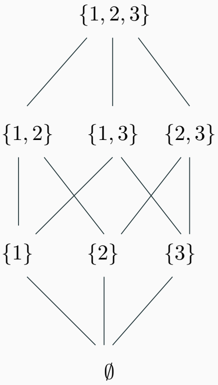

Powerset Poset

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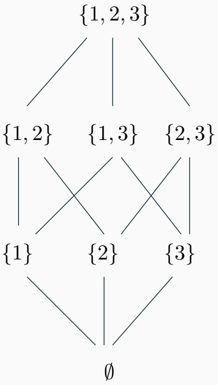

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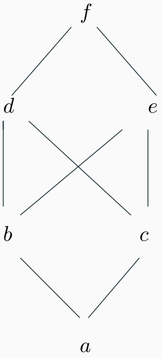

Poset

Lattices

- Given two elements a and b in a poset X, an **upper bound** is any element $c \in X$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$.
- Moreover, c is the least upper bound or **join** "$\sqcup$", if it exists is the smallest element greater than both a and b.
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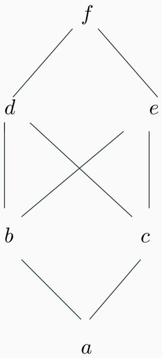
Definition (Lattice)

Lattices are particular posets which contain more structure. They are posets where the meet and join always exist for pairs of elements.

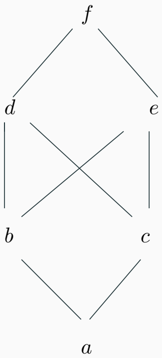
If the property holds for arbitrary set the lattice is said to be **complete**.

Lattice examples



Not a lattice, *d* and *e* do not have a meet.

Lattice examples



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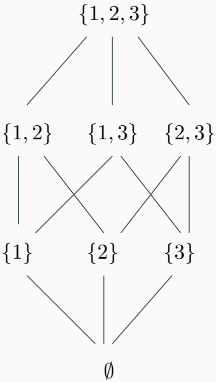
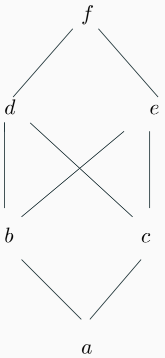


Powerset poset is a complete lattice

Lattice examples



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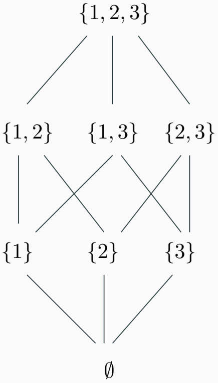

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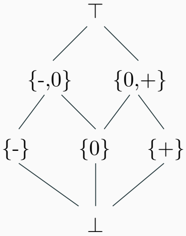

Sign poset is a complete lattice

Fixpoints

Fixpoints

Definition

Given a poset $(X, \sqsubseteq)$ and an operator $f : X \to X$:

- x is a **fixpoint** of f if $x = f(x)$. We denote as $fp(f) = \{x \in X | f(x) = x\}$ the set of fixpoints of f.
- x is a **prefixpoint** of f if $x \sqsubseteq f(x)$.
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- $lfp\, f = \min\{y \in fp(f)\}$ if it exists is called the **least fixpoint** of f.

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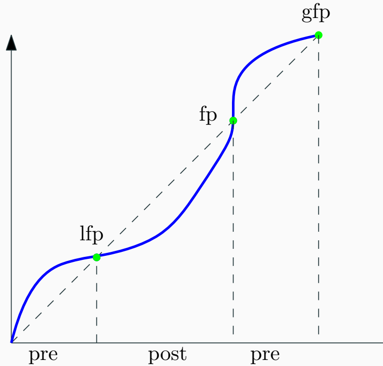
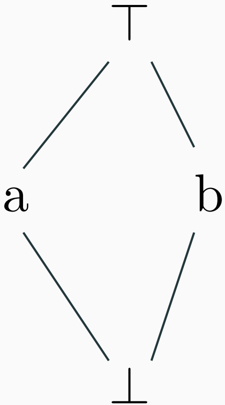


Fixpoint examples

Definition

$$f(\top) = \top$$
$$f(a) = a$$
$$f(b) = \top$$
$$f(\bot) = a$$

$$fp(f) = \{a, \top\}$$
$$lfp(f) = a$$
$$gfp(f) = \top$$
$$pre(f) = \{\bot, b\}$$
$$post(f) = \emptyset$$



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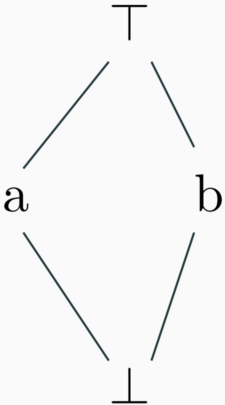

Knaster-Tarski Theorem

Definition

A function f in a complete lattice S is **monotone** if

$$\forall, x, y \in S, \quad x \sqsubseteq y \implies f(x) \sqsubseteq f(y)$$

Theorem ([T+55])

Let S be a complete lattice and f a monotone function on S then f admits a least fixpoint

$$\mathit{lfp}\, f = \bigsqcap \{x \in S, f(x) \sqsubseteq x\}.$$

Approximations

Concretization

When we are given a concrete and an abstract domain we need to be able to pass from one to another. This is done with two functions that are called **concretization** and **abstraction**.

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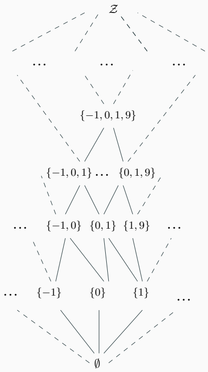
Example

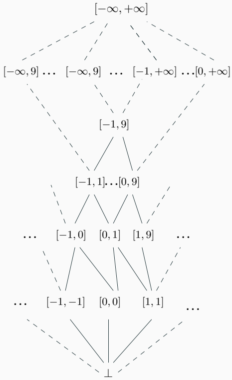
Let the concrete domain be the complete integer powerset lattice and the abstract domain be the interval lattice. Then elements of the abstract domain are intervals of the form $[a, b]$ with $a \geq b$. Then

$$\gamma([a, b]) = \{x \mid a \leq x \leq b\}$$

Concretization

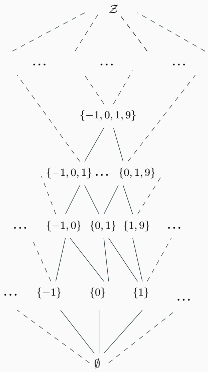


Powerset integer domain $\mathcal{C}$



Interval domain $\mathcal{A}$

Concretization



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The concretization function $\gamma : \mathcal{A} \to \mathcal{C}$ is defined by

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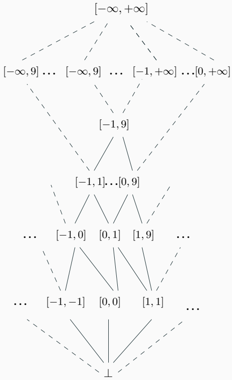
For example $\gamma([1,4]) = \{1,2,3,4\}$



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Abstraction

Definition (Sound abstraction)

Let $(A, \sqsubseteq), (C, \leq)$ be two posets. $a \in A$ is a **sound abstraction** of $c \in C$ if and only if $c \leq \gamma(a)$.

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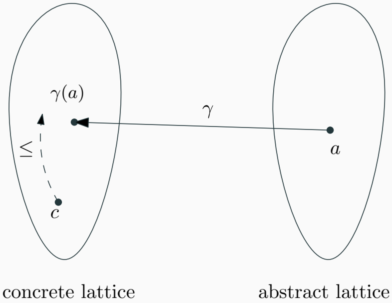

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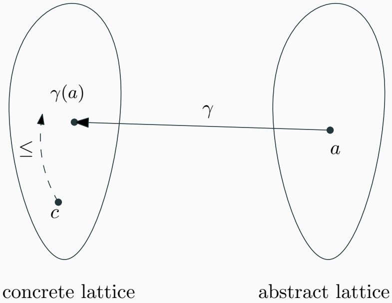

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With the same concrete and abstract domains abstraction function $\alpha : \mathcal{C} \to \mathcal{A}$ is defined by

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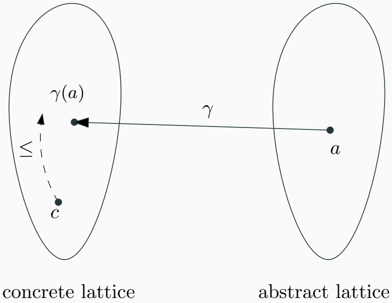

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Therefore $(\alpha \circ \gamma)(\{1,3,5\}) = \{1,2,3,4,5\}$ and $\{1,3,5\} \subseteq \{1,2,3,4,5\}$
We loose information when passing through the abstract domain but the result is sound (we get an overapproximation of the initial value).

Galois connection

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- more structure, if available, can help us design sound and accurate analyses.
- In the standard Abstract interpretation framework we assume additionally the existence of a monotonic abstraction function $\alpha : C \rightarrow A$ that associates an abstract element to a concrete one such that (α, γ) forms a Galois connection

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Given two posets $(A, \sqsubseteq)$ and $(C, \leq)$, the pair $(\alpha : C \to A,\ \gamma : A \to C)$ is a Galois connection if:

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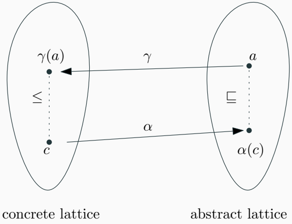

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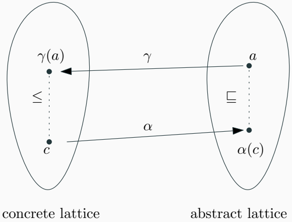
Properties of Galois connections:

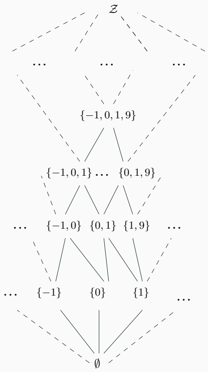
- $\gamma \circ \alpha$ is extensive : $\forall c, c \leq \gamma(\alpha(c))$.
- $\alpha \circ \gamma$ is extensive : $\forall a, \alpha(\gamma(a)) \sqsubseteq a$.
- α and γ are monotonic.

α and γ are called adjoint functions and each one of them can be defined in term of the other.

For instance $\alpha(c) = \sqcap\{a \mid c \leq \gamma(a)\}$



Galois connection examples



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The concretization function $\gamma : \mathcal{A} \rightarrow \mathcal{C}$ is defined by

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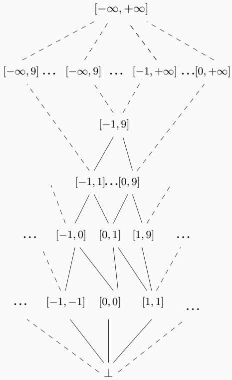

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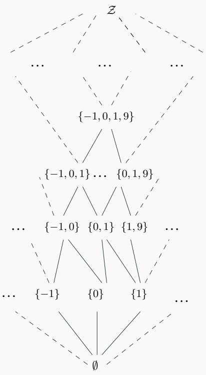
Galois connection examples



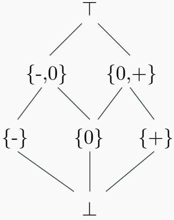
Powerset integer domain $\mathcal{C}$

The concretization function $\gamma : \mathcal{A} \to \mathcal{C}$ is defined by

$$\begin{aligned}
\gamma(\top) &= \mathbb{Z} \\
\gamma(\{-\}) &= \{x \mid x < 0\} \\
\gamma(\{+\}) &= \{x \mid x > 0\} \\
\gamma(\{-,0\}) &= \{x \mid x \leq 0\} \\
\gamma(\{0,+\}) &= \{x \mid x \geq 0\} \\
\gamma(\bot) &= \emptyset
\end{aligned}$$

and the abstraction function $\alpha : \mathcal{C} \to \mathcal{A}$ is defined by

$$\alpha(c) = \begin{cases} \{+\} \text{ if } c \in \mathbb{Z}_{>0} \\ \{-\} \text{ if } c \in \mathbb{Z}_{<0} \\ \{0,+\} \text{ if } c \in \mathbb{Z}_{\geq 0} \\ \{-,0\} \text{ if } c \in \mathbb{Z}_{\leq 0} \\ \bot \text{ if } c = \emptyset \\ \top \text{ else} \end{cases} \tag{1}$$



Abstract sign domain $\mathcal{A}$

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Definition (Sound operator abstraction)

Given a concretization γ from an abstract domain $(A, \sqsubseteq)$ to a concrete domain $(C, \leq)$, a concrete operator $f : C \rightarrow C$, and an abstract operator $g : A \rightarrow A$:
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$$c \leq \gamma(a) \Leftrightarrow \alpha(c) \sqsubseteq a$$

$$(f \circ \gamma)(a) \leq (\gamma \circ g)(a) \Leftrightarrow (\alpha \circ f \circ \gamma)(a) \sqsubseteq g(a).$$

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The lattice structure gives an automatic abstract operation

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Definition (Best operator abstraction)

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Example

Consider the Galois connection between powerset integer domain and interval domain. Let the operator $f : C \to C$ in the powerset domain be defined as $f(X) = \{x + 1 \mid x \in X\}$. $f(\{1, 3, 4\}) = \{2, 4, 5\}$.
The best abstraction of f in the abstract domain of intervals is thus defined by $f^\sharp = (\alpha \circ f \circ \gamma)$, then for example, $\alpha(\{1, 3, 4\}) = [1, 4]$ and $f^\sharp([1, 4]) = [2, 5]$.

Composition of best operator abstractions is not necessarily the best operator abstraction

That is if g and g' are the best abstractions of f and f' , then $(g \circ g')$ is not always the best abstraction!

Fixpoint transfer

Critical parts of the semantics of a program are defined as least fixpoints $\mathit{lfp}\, f$ of some monotonic or continuous operator $f : C \to C$ in the concrete domain $(C, \leq)$.
In order to abstract $\mathit{lfp}\, f$ in an abstract domain $(A, \sqsubseteq)$, a natural idea is to start with a sound abstraction $g : A \to A$ of f.

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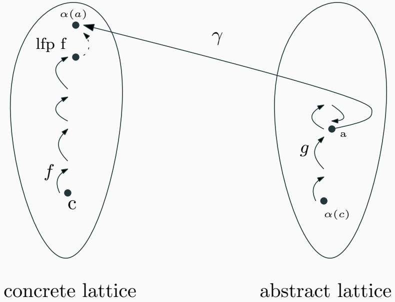
Theorem (Tarskian fixpoint transfer)

Given a complete lattice concrete domain, a monotonic concrete function $f : C \to C$, and a sound abstraction $g : A \to A$ in a poset abstract domain, then any postfixpoint a of g is a sound abstraction of lfp f, i.e lfp f $\leq \gamma(a)$



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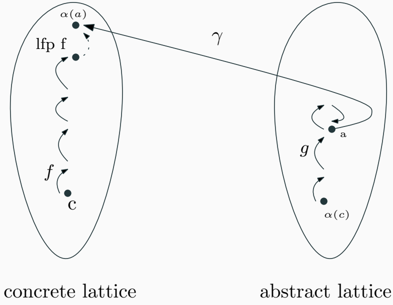
the theorem can be applied in the useful case where abstract fixpoints are hard to compute, or do not even exist at all. Sometimes fixpoints exist in the concrete, but are not guaranteed to exist in the abstract.



To keep in mind for next time

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- Order theory and lattices.
- Galois connections.
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Thank you for your attention

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