Aspects in Generating functions
Junior seminar

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Abstract

A very brief and smooth introduction to generating functions.

I made this notes by writing very simple ideas from some excellent books that deals in big parts with generating functions and more general topics in combinatorics. I highly recommend reading parts of them for any computer scientist or mathematician interested in the subject of average case analysis of algorithms, deep analysis of data structures, generating functions and combinatorics in general . [1],[2], [4] and [5].

1 The science of counting

> A generating function is a clothesline on which we hang up a sequence of numbers for display.
>
> (Herbert S. Wilf)

Suppose that we have a sequence of objects $a_0, a_1, a_2, \ldots, a_n; \ldots$ that we want to count. Most people would be happy with an explicit formula for the coefficients of the sequence. For example,

Example 1.

$$a_n = \begin{cases} 1, & n = 0 \\ \sum_{k=0}^{n-1} a_k & n \geq 1. \end{cases} \tag{1}$$

We see that $a_1 = 1$, $a_2 = 2$, $a_4 = 4$, $a_5 = 8$. *We can the deduce that the recurrence simplifies to* $a_n = 2a_{n-1}$ *and as a result,*

$$a_n = 2^{n-1}.$$

However, explicit formulas for the coefficients of a sequence are not always easy to find. For example, what can we say about?

Example 2.

$$f_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ f_{n-1} + f_{n-2} & n \geq 2. \end{cases} \tag{2}$$

The answer this time seems less obvious. Fortunately, this last sequence is the famous Fibonacci sequence and we have an explicit formula,

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - \phi^{-n}), \quad \phi = \frac{1+\sqrt{5}}{2}.$$

Which involves irrational numbers.

But for some people if asked about these examples. They would answer in a surprising way,

$$a_n = [z^n](1 + \frac{z}{1-2z}).$$

$$f_n = [z^n](\frac{1}{1-z-z^2}).$$

Where the operator $[z^n]G(z)$ is the coefficient of z^n in the expansion of $G(z)$. For example, if $G(z) = \frac{1}{1-3z} = \sum_{n\geq 0} 3^n z^n$. then,

$$[z^n]G(z) = [z^n]\sum_{n\geq 0} 3^n z^n = 3^n.$$

Moreover, they will call the functions $A(z) = (1+\frac{z}{1-2z})$ and $F(z) = (\frac{1}{1-z-z^2})$ the **generating functions** of a_n and f_n respectively.

Definition 1. *Let* a_n *be a sequence of numbers, its* ***ordinary generating function*** *is*

$$A(z) = \sum_{n\geq 0} a_n z^n.$$

Where z *will usually be a complex number.*

Ordinary generating functions are a type of generating functions where their kernel is equal to z^n. Other different types of generating functions exist.

$$\sum_{n\geq 0} a_n \frac{z^n}{n!} \quad \text{(Exponential generating function)}$$

$$\sum_{n\geq 0} a_n \frac{n!}{<t>_{n+1}} \quad \text{(Factorial generating function)}$$

$$\sum_{n\geq 0} a_n \frac{t^n}{1-t^n} \quad \text{(Lambert generating function)}$$

Where $<t>_n = t(t+1)\dots(t+n-1)$

It turned out that the generating functions are very powerful tools for different purposes. Here are some of the main that can be done with a generating function.

- Solving recurrences.
- Finding new recurrences.
- Getting asymptotics formulas for a recurrence (see [2] for a high quality exposition of existing methods and singularity theory).
- Proving identities.
- Showing proprorties of a sequence, unimodality, convexity,...

1.1 Computing a generating function from a sequence

General method:

1. Writing the recurrence of a_n.
2. Summing over all $n \geq 0$ and multiplying by z^n.
3. Equating both sides and solving for $A(z)$.

Example 3.

$$b_n = \begin{cases} 1, & n = 0 \\ 2a_{n-1} + (n-1) & n \geq 1. \end{cases} \tag{3}$$

The first terms of b_n *are,*

$$(b_n)_{n\geq 0} = 1, 2, 5, 12, 27, 58, 121, \dots$$

So we are interested in finding an expression for the generating function $B(z)$ of the sequence b_n,

$$B(z) = \sum_{n\geq 0} b_n z^n.$$

So we have,

$$b_{n+1} = 2b_n + n. \tag{4}$$

Summing the left-hand of (4) side over all $n \geq 0$ and multpilying by z^n gives

$$\sum_{n\geq 0} b_{n+1} z^n = \frac{1}{z}\sum_{n\geq 0} b_{n+1} z^{n+1} = \frac{1}{z}(B(z) - 1).$$

The right-hand side of (4) gives,

$$\sum_{n\geq 0} 2b_n z^n + \sum_{n\geq 0} n z^n = 2B(z) + \frac{z}{(1-z)^2}.$$

Now we are left with,

$$\frac{1}{z}(B(z) - 1) = 2B(z) + \frac{z}{(1-z)^2}.$$

Solving for $B(z)$ yields the desired generating function,

$$B(z) = \frac{1 - 2z + 2z^2}{(1-2z)(1-z)^2}.$$

Even though this expression does not seem to give much information about the sequence, a little manipulation of $B(z)$ gives a nice result. The idea is to decompose $B(z)$ in partial fraction,

$$B(z) = \frac{1 - 2z + 2z^2}{(1-2z)(1-z)^2} = \frac{A}{(1-2z)} + \frac{B}{(1-z)} + \frac{C}{(1-z)^2}. \tag{5}$$

Multiplying both sides of (5) by $(1-z)^2$ and replacing z=1 leads to,

$$-1 = C.$$

We can do the same thing by multiplying with $(1-2z)$ and find that,

$$2 = A.$$

And

$$B = 0.$$

Finally we get,

$$B(z) = \frac{2}{(1-2z)} + \frac{(-1)}{(1-z)^2}.$$

Extracting an exact formlua for b_n from the new generating is an easy task and we find that,

$$b_n = [z^n]\frac{2}{(1-2z)} + \frac{(-1)}{(1-z)^2} = 2^{n+1} - n - 1.$$

1.2 Operations on Ordinary Generating Functions

An important advantage of generating functions in variable z^n which are called **Ordinary generating functions** is that they behave nicely with basic operations. For instance,

$$\sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} b_n z^n = \sum_{n\geq 0} (a_n + b_n) z^n \qquad \text{(Disjoint union)}$$

$$(\sum_{n\geq 0} a_n z^n)(\sum_{n\geq 0} b_n z^n) = \sum_{n\geq 0} (\sum_{k=0}^{n} a_k b_{n-k}) z^n \qquad \text{(Cauchy product)}$$

$$\frac{d}{dz} \sum_{n\geq 0} a_n z^n = \sum_{n\geq 1} n a_n z^{n-1} \qquad \text{(Differentiation)}$$

A very useful generating function is the binomial expansion. We recall the result, which works for any positive integer k, but the result can be extended to non-integer values,

$$(1+z)^k = \sum_{i=0}^{n} \binom{n}{i} z^i = \sum_{i\geq 0} \binom{n}{i} z^i$$

1.3 Examples of identities

Example 4. *Show that,*

$$\sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n}.$$

At first sight this identity is not obvious even though a combinatorial meaning can be given to it. However, it can be shown easily using generating functions. We only need to recognize the fact $\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}$ *seems like a product of generating functions in* **??**.

$$[z^n](1+z)^a(1+z)^b = \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i}$$

But it is also,

$$(1+z)^a(1+z)^b = (1+z)^{a+b}.$$

Therefore,

$$[z^n](1+z)^{a+b} = \binom{a+b}{n}.$$

It is a very simple example of how combinatorial identities can be shown using generating functions. However, much more complicated ones can shown.

Exercice 1. *Show that,*

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$$

(Hint: Use Exponential generating functions, the basic operations are given in the Appendix 4)

Exercice 2. *Show that,*

$$\sum_{i=0}^{n} \binom{x+i}{i} = \binom{x+n+1}{n}$$

2 Analysis of Quicksort

Algorithm 1 Quicksort

function QUICKSORT(T,s,e)
    ▷ T is the array to be sorted and s, e the start and end indeces

    if $e \geq s$ **then**
        $v := T[e]$
        $i := s - 1$
        $j := e$
    while $i \geq T[i]$ **do**
        while $v \leq T[i]$ **do**
            $i := i + 1$
        while $v \geq T[j]$ **do**
            $j := j - 1$
        $tmp := T[i], \quad T[i] := T[j], \quad T[j] := tmp$
    $T[j] := T[i], \quad T[i] := T[e], \quad T[e] := tmp$
    Quicksort(T,e,$i-1$)
    Quicksort(T,$i+1$,s)

This algorithm was invented by C. A. R. Hoare in 1961 [3] who did one the first analysis of algorithms. More details can be found in [1].

The algorithm partitions the array into two parts, following the last element of the array $T[e]$. The first partition will contain all elements smaller than $T[e]$ and the second all elements that are larger than $T[e]$.Places $T[e]$ in its right place and finally, the algorithm is called back on the two partitions independently.

The algorithm Quicksort(T,1,n) will sort an array of n elements, but we need an additional element $T[0]$ that has to be smaller than all others.

Exercice 3. *Let $T = [-1, 2, 4, 3, 1]$, Compute Quicksort(T,1,4).*

The average number of comparison made by Quicksort is

$$c_n = \begin{cases} 0, & n = 0 \\ (n+1) + \frac{1}{n}\sum_{i=1}^{n}(c_{i-1} + c_{n-i}) & n \geq 1. \end{cases} \tag{6}$$

Here we consider that the elements of the table form a uniform random permutation. However other distribution models can be studied too in the same way.

Multiplying both side of 6 by n we get the following,

$$nc_n = n(n+1) + 2\sum_{i=1}^{n} c_{i-1}. \tag{7}$$

We want to find the generating function of c_n

$$C(z) = \sum_{n\geq 0} c_n z^n.$$

We can use the same approach as before, after summing for all $n \geq 0$ and multiplying by z^n we get,

$$\sum_{n\geq 0} nc_n z^n = \sum_{n\geq 0} n(n+1)z^n + 2\sum_{n\geq 0}\sum_{i=1}^{n} c_{i-1}z^n. \tag{8}$$

The left-hand side of (8) is equal to

$$zC'(z)$$

For the right-hand side of (8) we notice that $\sum_{n\geq 0}\sum_{i=1}^{n} c_{i-1}z^n$ is a Cauchy product of $\frac{1}{(1-z)}$ and $zC(z)$. Therefore,

$$\sum_{n\geq 0}\sum_{i=1}^{n} c_{i-1}z^n = \frac{z}{(1-z)}C(z)$$

Exercice 4. *Show that,*

$$\sum_{n\geq 1} n(n+1)z^n = \frac{2z}{(1-z)^3}.$$

Finaly the right-hand side gives,

$$\frac{2z}{(1-z)^3} + 2\frac{z}{(1-z)}C(z)$$

Then we have $zC'(z) = \frac{2z}{(1-z)^3} + 2\frac{1}{(1-z)}zC(z)$. Which simplifies to

$$C'(z) = \frac{2}{(1-z)^3} + 2\frac{1}{(1-z)}C(z). \tag{9}$$

The differential equation can be solved in simple steps. a detailed proof is given in the Appendix(4),

$$C(z) = \frac{2}{(1-z)^2}\ln\frac{1}{(1-z)}.$$

Theorem 1. *The average number of comparisons in Quicksort is*

$$c_n \sim n \ln n.$$

Proof. To see this we only need to extract coefficients from $C(z)$ and take the first order asymptotics of H_n.

$$\begin{aligned}[z^n]C(z) =& 2\sum_{i=1}^{n}(\frac{1}{i})(n-i+1)\\ =& 2(nH_{n+1} - n + H_{n+1})\\ =& 2((n+1)H_{n+1} - n).\end{aligned}$$

Where $H_n = \sum_{i=1}^{n} \frac{1}{i}$ is the sequence of **Harmonic numbers**. The asymptotics of H_n can be computed via the theory of Analytic Combinatorics exposed in [2]. and we have,

$$H_n \sim \ln n + \gamma$$

□

3 Glimpse to singularity analysis

In the next example we will show how generating functions can give surprisingly easy method to solve seemingly complicated problems. This time we will use Exponential generating function.

Example 5. ***2-regular graphs***. *A 2-regular graph is a graph in which each vertex has degree* 2. *We want to count 2-regular graphs on n vertices. The vertices are labelled, this explains why it is more convenient to use Exponential generating functions. We want,*

$$R(z) = \sum_{n \geq 0} r_n \frac{z^n}{n!}$$

Thinking of this problem we can see that these graphs are made of disjoint undirected cycles. And the number of disjoint undirected cycles on n vertices is $\frac{(n-1)!}{2}$. Therefore we get,

$$\begin{aligned}D(z) =& \sum_{n \geq 3} \frac{(n-1)!}{2} \frac{z^n}{n!}\\ =& \frac{1}{2} \sum_{n \geq 3} \frac{z^n}{n}\\ =& \frac{1}{2}\left(\log \frac{1}{1-z} - z - \frac{z^2}{2}\right)\end{aligned}$$

$D(z)$ is the EGF of the counting sequence of the graphs so finally,

$$\begin{aligned} R(z) =& e^{D(z)} \\ =& \frac{e^{\frac{-z}{2} - \frac{z^2}{4}}}{\sqrt{1-z}}. \end{aligned}$$

The theory of singularity analysis then tells us that expanding the numerator around $z = 1$ gives,

$$R(z) = \frac{e^{-\frac{3}{4}}}{\sqrt{1-z}} + O((1-z)^{\frac{1}{2}}).$$

And then,

$$r_n \sim n! \frac{e^{-\frac{3}{4}}}{\sqrt{\pi n}}$$

For the complete study see [2, p. 379]

4 Appendix

Solving the differential equation (9),
The homogenous equation has the form,

$$\frac{f'(z)}{f(z)} = \frac{2}{(1-z)}.$$

Which gives, $f(z) = \frac{1}{(1-z)^2}$ as an integration factor. Now,

$$\begin{aligned} ((1-z)^2 C(z))' =& C'(z)(1-z)^2 - 2(1-z)C(z) \\ =& (1-z)^2 (C'(z) - 2\frac{C(z)}{(1-z)}) = \frac{2}{(1-z)}. \end{aligned}$$

Operations on Exponential Generating Functions
As in the case of ordinary generating functions the exponential ones have also simple forms for basic operations. For instance,

$$\sum_{n \geq 0} a_n \frac{z^n}{n!} + \sum_{n \geq 0} b_n \frac{z^n}{n!} = \sum_{n \geq 0} (a_n + b_n) \frac{z^n}{n!} \qquad \text{(Disjoint union)}$$

$$(\sum_{n \geq 0} a_n \frac{z^n}{n!})(\sum_{n \geq 0} b_n \frac{z^n}{n!}) = \sum_{n \geq 0} (\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}) \frac{z^n}{n!} \qquad \text{(Cauchy product)}$$

$$\frac{d}{dz} \sum_{n \geq 0} a_n \frac{z^n}{n!} = \sum_{n \geq 1} a_n \frac{z^{n-1}}{(n-1)!} \qquad \text{(Differentiation)}$$

References

[1] Philippe Flajolet and Robert Sedgewick. *An introduction to the analysis of algorithms*. Addison-Wesley-Longman, 1996.

[2] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, New York, NY, USA, 1 edition, 2009.

[3] Charles A. R. Hoare. Algorithm 64: Quicksort. *Commun. ACM*, 4(7):321–, July 1961.

[4] Richard P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, New York, NY, USA, 2nd edition, 2011.

[5] Herbert S. Wilf. *Generatingfunctionology*. A. K. Peters, Ltd., Natick, MA, USA, 2006.