Combinatorics of increasing trees:

Bijections, asymptotics and algorithms

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Under the supervision of
Olivier Bodini and Antoine Genitrini
In fulfilment of the degree Doctor of Philosophy of
Université Sorbonne Paris-Nord

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Julien Clément	Université de Caen
Alois Panholzer	Technische Universität Wien

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18 December 2020

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Introduction

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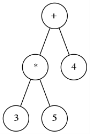

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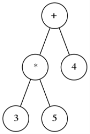
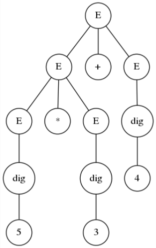


Abstract syntax tree



Its corresponding parse tree

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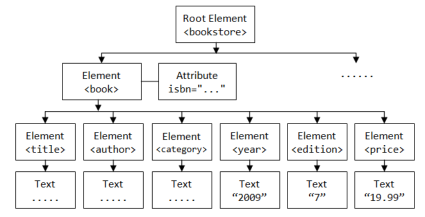
- Markup languages such as XML that have underlying tree structures that can be used and processed subsequently by the *Document Object Model*.



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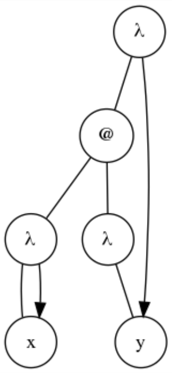
- Lambda terms in lambda calculus are enriched tree structures.



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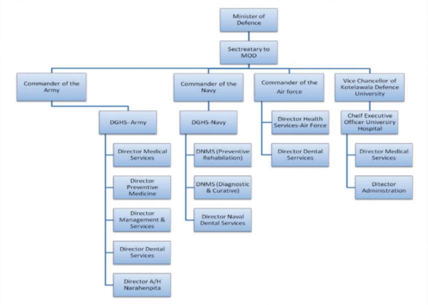

The contemporary structure of the Sri Lankan military health care services. (military-medicine)

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- In biology and phylogenetics to represent the evolutionary relationship among species. [Fel03, Ste16]

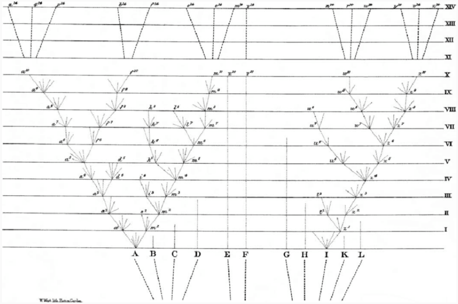

Diagram of divergence of Taxa 1871. Darwin (On the origin of species)

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Eukaryotes Tree of Life 2020, showing positions of fungi and fungus-like organisms. Tricholome (wikipedia)

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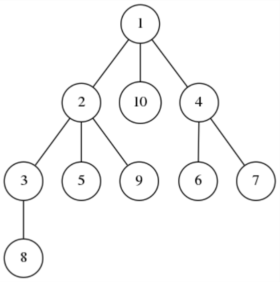

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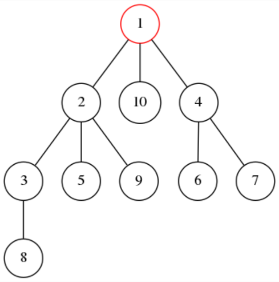

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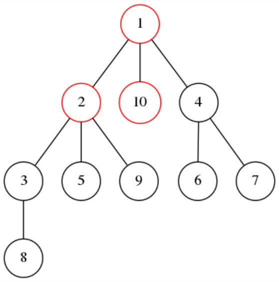

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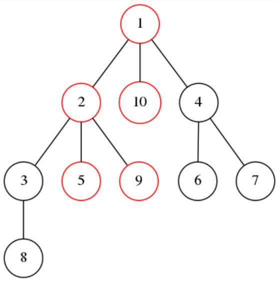

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AW: accumulated weight of singular readings
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TN: indication whether terminal (T) or not (N)

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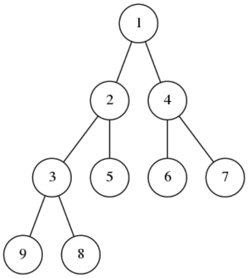

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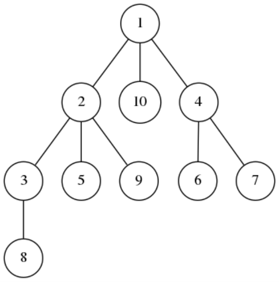

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- Study the number of executions of a parallel process and their synchronisations. This leads to repeated labellings. [BGP16, BGR17]

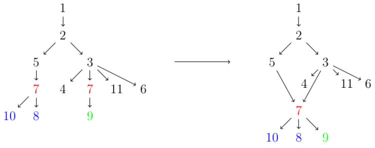

Figure 1. A synchronized tree

From [BGR17]

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Both approaches are complementary. It is possible to study random trees and derive similar type of results.

Analytic combinatorics

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Framework to analyse properties of large random structures belonging to specifiable combinatorial classes

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- **Symbolic method** developed in [FS09] is a grammar used to define (specify) combinatorial classes:
  1. Elementary constructions are the neutral class and the atomic class.
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Result

Operations in the symbolic method translates directly to operations on generating functions.

- Theorems for automatic asymptotic estimates.
- Theorems for the shape of large random structures.

Ordinary generating functions

For a combinatorial class $\mathcal{C}$ we define its *ordinary generating function* (OGF) to be $C(z) = \sum_{n=0}^{\infty} C_n z^n$ where C_n counts the number of objects in $\mathcal{C}$ of size n.

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Example: Words over the 3 letter alphabet $\{{\color{blue}\bullet}, {\color{green}\bullet}, {\color{red}\bullet}\}$ For example $w = {\color{red}\bullet}{\color{green}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{green}\bullet}{\color{blue}\bullet}{\color{blue}\bullet}{\color{green}\bullet}{\color{blue}\bullet}{\color{green}\bullet}{\color{red}\bullet}{\color{blue}\bullet} \in \mathcal{W}$

$$\mathcal{W} = \text{SEQ}\,(\mathcal{Z}_{\color{blue}\bullet} + \mathcal{Z}_{\color{green}\bullet} + \mathcal{Z}_{\color{red}\bullet}) \overset{\text{Symbolic method}}{\Longrightarrow} W(z) = \frac{1}{1 - 3z}$$

The terms are then obtained by coefficient extraction $[z^n]W(z)$
$W(z) = 1 + 3z + 9z^2 + 27z^3 + 81z^4 + 243z^5 + 729z^6 + 2187z^7 + 6561z^8 + 19683z^9 + 59049z^{10} + \ldots$

Plane simple trees varieties

Plane simple trees are rooted unlabelled trees

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Definition (Weighted degree function)

For a class of trees with ϕ_i colours for i-ary nodes, we define its degree function to be $\phi(z) = \sum_{i\geq 0} \phi_i z^i$.

For example

$$\phi(z) = 1 + z^2 + 10z^3 + 2z^5.$$

Corresponds to a class of trees having:

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Given a weighted degree function $\phi(z)$ such that $\phi_0 > 0$, the variety of plane simple trees parameterised by ϕ is specified in world of OGF by,

$$\mathcal{T} = \mathcal{Z} \times (\phi \circ \mathcal{T})$$

which gives,

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Binary trees are parameterised by

$\phi(z) = 1 + 2z + z^2$, then,

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In plane simple trees nodes can be decorated but do not bear labels

Exponential generating functions

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Example: Ordered set partitions. $[\{2, 4, 5\}, \{1, 7\}, \{3, 6\}]$ is an ordered partition of size 7.

$$\mathcal{B} = \textsc{Seq}\left(\textsc{Set}_{\geq 1}(\mathcal{Z})\right) \overset{\text{symbolic method}}{\Longrightarrow} B(z) = \frac{1}{1 - (\exp(z) - 1)} = \frac{1}{2 - \exp(z)}$$

The terms are then obtained by $n![z^n]B(z)$ and are called Ordered Bell numbers:
$B(z) = 1z + 3z^2 + 13z^3 + 75z^4 + 541z^5 + 4683z^6 + 47293z^7 + 545835z^8 + \ldots$

Greene operator and increasing trees

- Labelled structures are naturally specified with EGF since each atom bears an integer label. Then the normalisation $\frac{z^n}{n!}$ insures the generating function to be convergent and analytic methods apply.
- The term increasing trees classically refers to trees such that the labels are strictly increasing along branches and have no label repetitions.[BFS92]

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$$\mathcal{B} = \epsilon + \mathcal{Z}^{\square} \star (\mathcal{B} \star \mathcal{B}) \stackrel{\text{symbolic method}}{\Longrightarrow} B(z) = 1 + \int_0^z 1 \cdot B^2(t)\, dt$$

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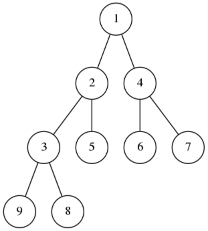
which solves to

$$B'(z) = B^2(z), B(0) = 1 \implies B(z) = \frac{1}{1-z}$$

$$B_n = n![z^n]B(z) = n!.$$



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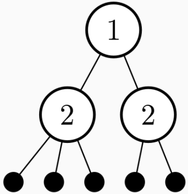

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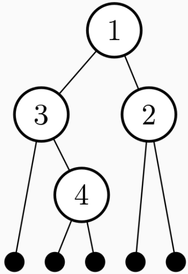

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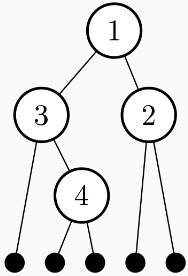

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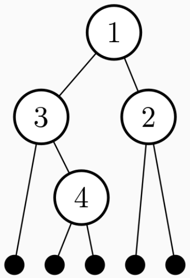
Can be modeled using trees such that

- Internal nodes bear integer labels corresponding to the time of differentiation (label repetitions are allowed).
- The size of the tree is its number of leaves.
- Nodes can have different arities.
- Branches are strictly increasing (label repetitions allowed).



Parametrisable evolution process for classes of strict monotonic Schröder trees

Parameters of the evolution process

Definition (Coloured degree function)

For a class of trees with ϕ_i colours of i-ary nodes, we define its coloured degree function to be $\phi(z) = \sum_{i \geq 1} \phi_i z^i$.

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The set $r \subset \mathbb{N}^*$.

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$$\phi(z) = z^2 + 10z^3 + 2z^5.$$

Corresponds to a class of trees having:

- Binary nodes of 1 colour.
- Ternary nodes of 10 colours.
- 5-ary nodes of 2 colours.

A coloured degree function is a weighted degree function where $\phi_0 = 0$.

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$$r = \{2, 3, 5\}.$$

At each iteration step there are either 2, 3 or 5 repetitions of the same label (i.e the number of leaves that evolves at each step is constrained to lie in r).

Evolution process for a variety of strict monotonic trees

The evolution process has two parameters a coloured degree function and a set of allowed repetitions.

$$\phi(z) = 2z^2 + 2z^3 \quad \text{and } r = \mathbb{N}^*$$

We have two colours of binary nodes (w,g) and two colours of ternary nodes (w,g).

Start at step 0 with a leaf; at each step $i \geq 1$ do:

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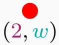

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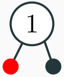

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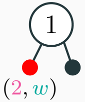

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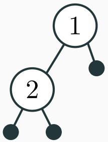

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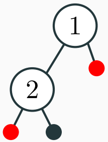

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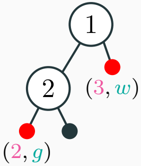

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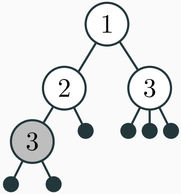

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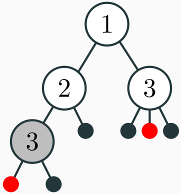

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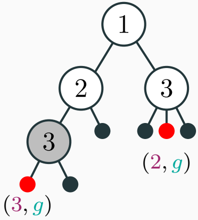

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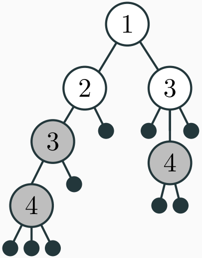

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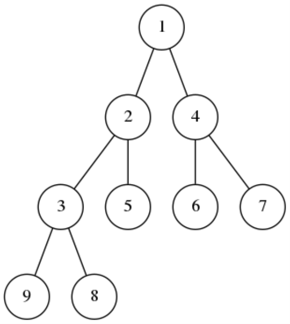
Which translates to,

$$B(z) = z^m + \sum_{i \in r} \frac{1}{i!} B^{(i)}(z) \left(\phi(z)^i - (\phi_1\, z)^i\right).$$

$\frac{B^{(i)}(z)}{i!}$ corresponds to erasing i leaves.

Related works on increasing trees

- Increasing trees [BFS92].
  No label repetitions and labellings along branches are strictly increasing.



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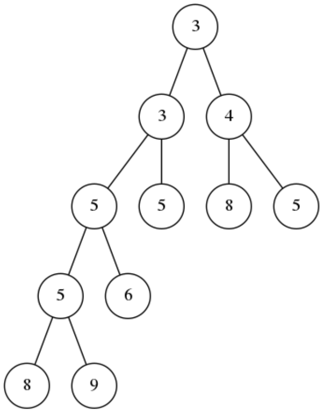

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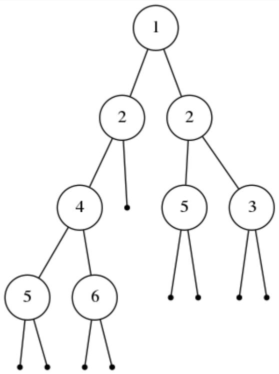

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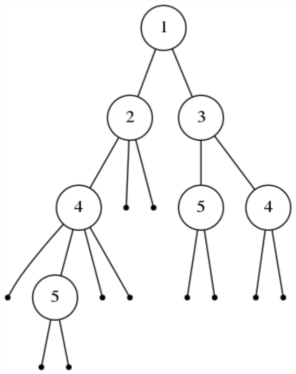

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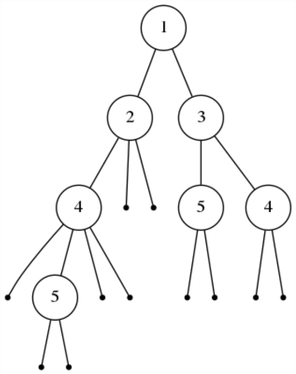
- Families of monotonic trees [BGNS20].
  A general asymptotic for cases where the number of repetitions allowed is not bounded.



Tree classes built with this evolution process

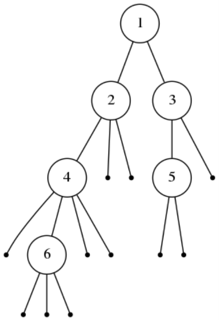
r	$\phi(z)$	Name	References
$\{1\}$	z^d	Plane d-ary increasing	[BFS92]
$\{1\}$	$\frac{z^2}{1-z}$	Increasing Schröder	[BGN19]
$\mathbb{N}^*$	z^2	Strict monotonic binary	[BGGW20]
$\mathbb{N}^*$	$\frac{z^2}{1-z}$	Strict monotonic Schröder	[BGN19]
$\mathbb{N}^*$	$\frac{z}{1-z}$	Strict monotonic general Schröder	[BGMN20]
$\mathbb{N}^*$	plane d-ary	Monotonic d-ary trees	[BGNS20]
$\{1, 2\}$	z^2	Supertrees	[SDH$^+$04]
$\{d\}$	z^2	Increasing binary with d label repetitions	

Table 1: Some of examples of tree classes covered by the evolution process

Three particular cases of the evolution process

Increasing Schröder trees

Corresponds to the parameters $\phi(z) = \frac{z^2}{1-z}$ and $r = \{1\}$. Therefore, a single leaf can evolve at each iteration step.



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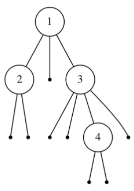
- Number of trees is $\frac{n!}{2}$.
- Bijections with half permutations that preserves several parameters. Number of internal nodes and the depth of the leftmost leaf are related to the the number of cycles in a permutation.



This tree with 8 leaves Corresponds to the permutation $(1, 4, 5)(2)(3)(6, 8)(7)$ it has $8 + 1 - 5 = 4$ internal nodes.

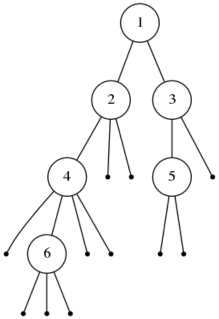
Increasing Schröder trees

Corresponds to the parameters $\phi(z) = \frac{z^2}{1-z}$ and $r = \{1\}$. Therefore, a single leaf can evolve at each iteration step.

- Number of trees is $\frac{n!}{2}$.
- Bijections with half permutations that preserves several parameters. Number of internal nodes and the depth of the leftmost leaf are related to the the number of cycles in a permutation.
- Typical parameters are:

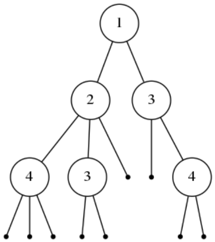
	Mean	Variance	Limit law
Internal nodes	$n - \ln n$	$\ln n$	Normal
Depth of the leftmost leaf	$\ln n$	$\ln n$	Normal
Height of the tree	$\Theta(\ln n)$		
Degree of the root	$2\,\mathrm{e} - 3$	$14\,\mathrm{e} - 4\,\mathrm{e}^2 - 8$	Modified Poisson

	2-ary	3-ary	4-ary	5-ary	6-ary	7-ary	8-ary	9-ary	10-ary
$\mathbb{E}C_n^{(\ell)}$	$n - 2\ln n$	$\ln n$	$\frac{23}{90}$	$\frac{1}{32}$	$\frac{107}{25200}$	$\frac{47}{86400}$	$\frac{101}{1587600}$	$\frac{229}{33868800}$	$\frac{659}{1005903360}$



Strict monotonic Schröder trees

Corresponds to the parameters $\phi(z) = \frac{z^2}{1-z}$ and $r = \mathbb{N}^*$. Any number of repetitions is allowed and the node degrees can be anything ≥ 2.



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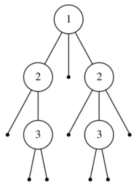

This tree with 8 leaves Corresponds to the ordered set partition $(\{3,4\},\{1,5,7\},\{2,6\})$. The tree has 3 distinct labels and the partition 3 subsets.

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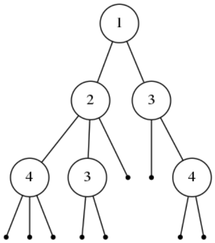
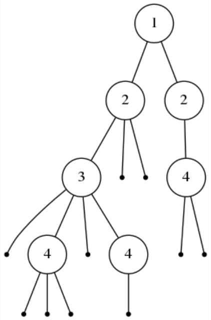
	Mean	Variance	Limit law
Internal nodes	$n - \ln 2 \ln n$		
Distinct labels	$\frac{1}{2\ln 2} n$	$\frac{(1-\ln 2)}{(2\ln 2)^2} n$	Normal
Degree of the root	$2\ln 2 + 1$	$-2\ln 2(\ln 2 - 1)$	Shifted zero-truncated Poisson
Depth of the leftmost leaf	$\ln n$	$\ln n$	Normal



Strict monotonic general Schröder trees

Corresponds to the parameters $\phi(z) = \frac{z}{1-z}$ and $r = \mathbb{N}^*$. Any number of repetitions is allowed and the node degrees can be anything ≥ 1 **including unary nodes**.



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- $f_n \underset{n\to\infty}{=} c\, 2^{\frac{(n-1)(n-2)}{2}}\, (n-1)!$.
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	Mean
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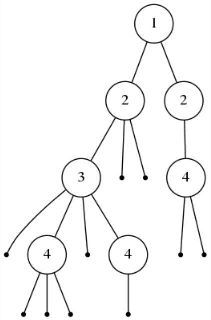

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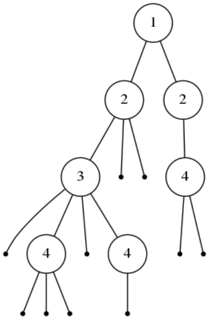


Remark

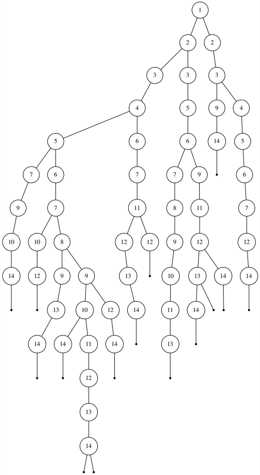
Unary nodes change significantly the number of trees and its typical shape.

Summary on the three models

	Number of trees	Number of internal nodes
Increasing Schröder trees	$n!/2$	$n - \ln n$
Strict monotonic Schröder trees	$(n-1)!/(2\,(\ln 2)^n)$	$n - \ln 2\ \ln n$
Strict monotonic general trees	$c\ 2^{(n-1)(n-2)/2}\ (n-1)!$	$\Theta(n^2)$

Remark

For all three models, it seems that typical large trees have nodes that are mostly from the first allowed arity, and little from the second allowed arity while other arities appear very rarely.



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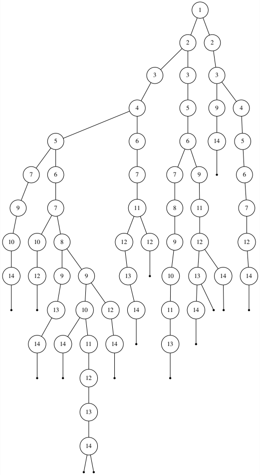

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For all three models, it seems that typical large trees have nodes that are mostly from the first allowed arity, and little from the second allowed arity while other arities appear very rarely.

This idea is developed in the next theorems in the form of asymptotic enumeration.

Main theorem I

Condition

Let $r \subset \mathbb{N}^*$ and $m = min(r)$. Let $\phi(z)$ be a *coloured degree function* and such that $\phi_1 = 0$, $\phi_2 \geq 1$ and $\phi_n \underset{n \to \infty}{=} O\left(\frac{n!}{m!^{n/m}\ n^{m+4}}\right)$.

Let $B_n^{\phi,r}$ be the number of trees of size n built via the evolution process

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Theorem

Let $\phi(z)$ be a coloured degree function as in the condition, and $r \subset \mathbb{N}^*$, with $r \neq \emptyset$. Let $m = min(r)$, then as n tends to infinity and is of the form $n \equiv 0 \mod m$,

$$B_n^{r,\phi} \underset{n\to\infty}{\sim} \kappa\ n!\ \left(\frac{\phi_2}{\rho}\right)^n\ n^{-1+\frac{\rho\,\phi_3}{\phi_2^2}-\frac{\rho\, f''(\rho)}{f'(\rho)}},$$

where κ is a constant that depends on $\phi(z)$ and r. Let $f(z) = \sum_{i\in r} \frac{z^i}{i!}$, then ρ is the smallest positive real of the equation $f(z) - 1 = 0$.

Main theorem II

Condition

Let $\phi(z)$ be a *coloured degree function* and such that $\phi_1 \geq 1$, $\phi_2 \geq 1$ and $\phi_n = O\left(\frac{n!}{n^5}\right)$.

Let $B_n^{\phi,r}$ be the number of trees of size n built via the evolution process

Main theorem II

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Let $\phi(z)$ be a *coloured degree function* and such that $\phi_1 \geq 1$, $\phi_2 \geq 1$ and $\phi_n = O\left(\frac{n!}{n^5}\right)$.

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Theorem

Let $\phi(z)$ be a coloured degree function as in the condition, let $r \subset \mathbb{N}^*$, $r \neq \emptyset$, and $r \neq \{1\}$, then as n tends to infinity,

$$B_n^{r,\phi} \underset{n\to\infty}{\sim} \kappa\,(n-1)!\,\phi_2^{n-1} \prod_{k=1}^{n-1} \left(\sum_{i=1, i\in r}^{n-k} \phi_1^{i-1} \binom{n-k-1}{i-1} \right),$$

where κ is a constant that depends on $\phi(z)$ and r.

The condition on ϕ_2 can be relaxed and a similar result holds.

Applications

Increasing binary trees with d repetitions

- $\mathcal{B}^d$ be the class of increasing binary trees with d label repetitions at each iteration step.
- At each iteration step exactly d leaves are chosen to expand (we start with a single root that has d leaves).

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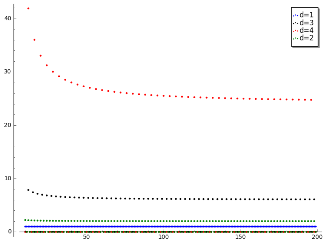
- At each iteration step exactly d leaves are chosen to expand (we start with a single root that has d leaves).
- Specification with $\phi(z) = z^2$ and $r = \{d\}$.

d	Asymptotics	References
1	$(n-1)!$	EIS A000142
2	$c_2\, n!\, \left(2^{1/2}\right)^{-n} n^{-2}$	EIS A000680
3	$c_3\, n!\, \left(3!^{1/3}\right)^{-n} n^{-3}$	EIS A014606
4	$c_4\, n!\, \left(4!^{1/4}\right)^{-n} n^{-4}$	EIS A014608

Table 2: Asymptotic behaviour for B_n^d for $d \in \{1,2,3,4\}$ when $n \equiv 0 \mod d$. The sequences in OEIS appear shifted (without periodicities).



Simulation for $n \in \{10, 200\}$ of B_n^d divided by their expected asymptotic behaviour with $d \in \{1,2,3,4\}$.

Evolution process for monotonic trees

Example: Monotonic binary trees

Take $\phi(z)$ to be unlabelled binary trees (or Catalan trees)

$$\phi(z) = cat(z) - z = z^2 + 2z^3 + 5z^4 + \dots \quad \text{and } r = \mathbb{N}^*$$

Start at step 0 with a leaf; at each step $i \geq 1$ do:

1. Choose a non-empty subset L of leaves of the so-far built tree such that $|L| \in r$.
2. For each $\ell \in L$ choose an integer $k > 1$ such that $\phi_k > 0$, and one of the $1 \leq c \leq \phi_k$ possible unlabelled trees with k leaves.
3. Replace ℓ with the chosen unlabelled tree, labelling all its internal nodes i.

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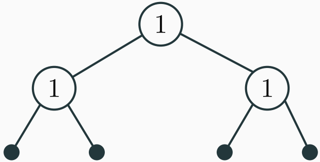

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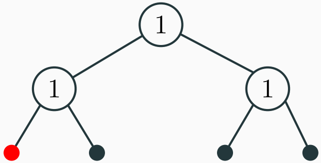

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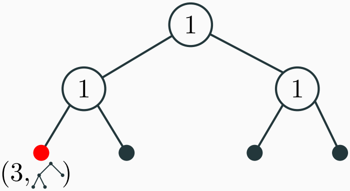

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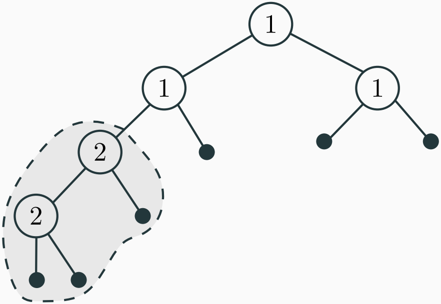

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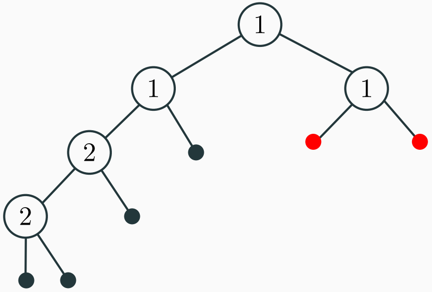

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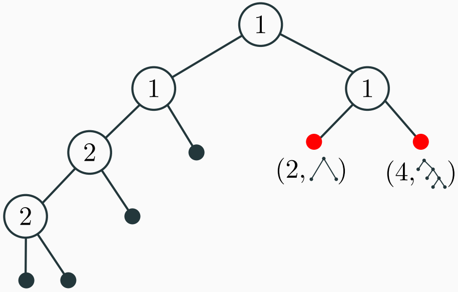

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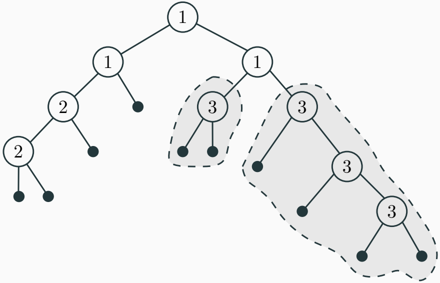

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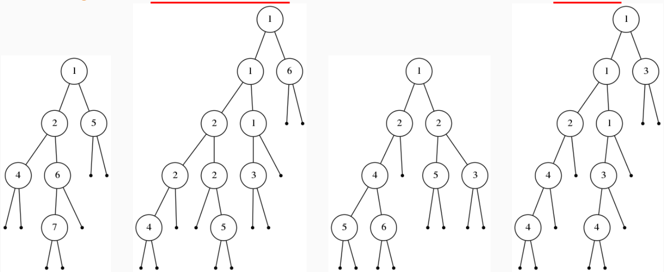
Increasing labellings of binary trees

As a result we propose new **increasing labellings models** on trees which allow label repetitions:



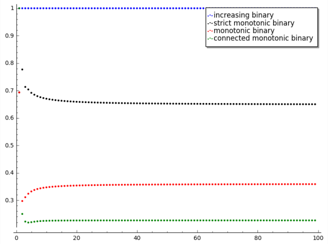
	Increasing	Connected monotonic	Strict monotonic	Monotonic
Label repetitions	no	yes	yes	yes
Branches	strictly increasing	weakly increasing	strictly increasing	weakly increasing
Repetitions		in the same subtree	anywhere	anywhere

- Using different values of ϕ and r we can specify and enumerate trees (counted by their leaves) with the different increasing labellings.
- Theorem I applies to give the asymptotic behaviours.

Comparison of binary trees increasing labellings

	r	$\phi(z)$	Asymptotics	References
Increasing	$\{1\}$	z^2	$(n-1)!$	[FS09], Theorem I
Connected monotonic	$\{1\}$	$(cat(z)-z)$	$c_3\, n!\, n$	Theorem I
Strict monotonic	$\mathbb{N}^*$	z^2	$c_4\,(n-1)!\,(\frac{1}{\ln 2})^n\, n^{-\ln 2}$	[BGGW20], Theorem I
Monotonic (weakly increasing)	$\mathbb{N}^*$	$(cat(z)-z)$	$c_5\,(n-1)!\,(\frac{1}{\ln 2})^n\, n^{\ln 2}$	Theorem I

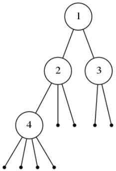
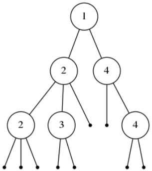
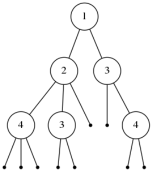
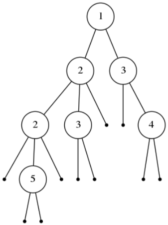
Table 3: Comparison of the asymptotic behaviour of labelled binary trees under different labelling models.



Simulation for $n \in \{1, 100\}$ of binary trees with different increasing labellings divided by their expected asymptotic first order.

Increasing labellings of Schröder trees

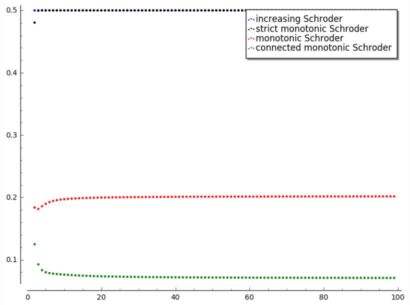
On **Schröder trees** the different increasing labellings give:

	Increasing	Connected monotonic	Strict monotonic	Monotonic
				
Label repetitions	no	yes	yes	yes
Branches	strictly increasing	weakly increasing	strictly increasing	weakly increasing
Repetitions		in the same subtree	anywhere	anywhere

Comparison of Schröder trees under different increasing labellings

	r	$\phi(z)$	Asymptotics	References
Increasing Schröder trees	$\{1\}$	$\frac{z^2}{1-z}$	$\frac{1}{2}\, n!$	[BGN19], Theorem I
C. M. Schröder trees	$\{1\}$	$(S(z)-z)$	$\alpha\ n!\ n^2$	Theorem I
Strictly monotonic Schröder	$\mathbb{N}^*$	$\frac{z^2}{1-z}$	$\frac{1}{2}(n-1)!\ (\frac{1}{\ln 2})^n$	[BGN19], Theorem I
Monotonic Schröder	$\mathbb{N}^*$	$(S(z)-z)$	$\kappa(n-1)!\ (\frac{1}{\ln 2})^n\ n^{2\ln 2}$	Theorem I

Table 4: Comparison of the asymptotic behaviour of families of labelled Schröder trees. $S(z)$ is the GF of Schröder trees.



Simulation for $n \in \{1, 100\}$ of Schröder trees with different increasing labellings divided by their expected asymptotic first order.

Conclusion and future works

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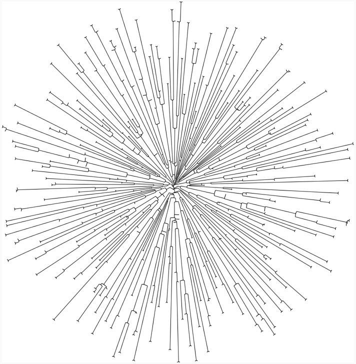

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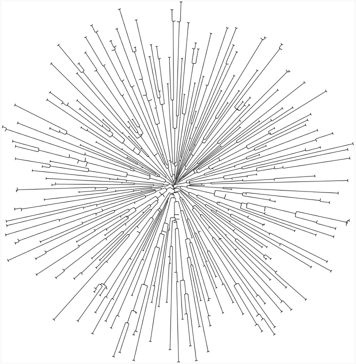

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- We applied it on increasing binary trees and get the average compaction rate is $\Theta\left(\frac{n}{\ln n}\right)$ and on recursive tree to get $O\left(\frac{n}{\ln n}\right)$.

Publications and preprints

O. Bodini, A. Genitrini, M. Naima	"Ranked Schröder trees". In Proceedings of the Sixteenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO), 2019.
O. Bodini, A. Genitrini, M. Naima, A. Singh	"Families of Monotonic Trees: Combinatorial Enumeration and Asymptotics". In Proceedings of the 15th International Computer Science Symposium in Russia (CSR), 2020.
O. Bodini, A. Genitrini, C. Mailler, M. Naima	"Strict monotonic trees arising from evolutionary processes: combinatorial and probabilistic study". Submitted to a journal. Available on https://hal.sorbonne-universite.fr/hal-02865198
O. Bodini, A. Genitrini, B. Gittenberger, I. Larcher, M. Naima	"Compaction for two models of logarithmic-depth trees: Analysis and Experiments". Submitted to a journal. Available on https://arxiv.org/abs/2005.12997

Perspectives and future works

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Increasing labellings
We obtained results on different increasing labellings of tree structures counted by their number of leaves. What about trees in which we count all nodes as it is mostly the case?

Thank you

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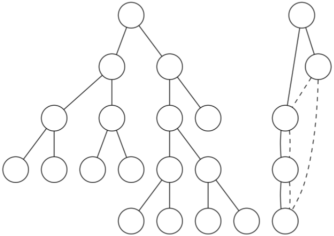
Compaction of trees

- Another interesting equation related to trees is their optimal representation memory.



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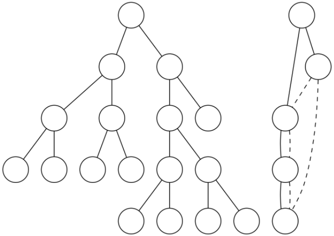
- The idea is that in a single tree, some subtrees can be isomorphic and therefore when compressing the tree we can keep only one occurrence of a repeated subtree and put pointers to it.



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- We found a generic method to obtain the average compaction rate of classes of increasing trees based on a perturbed generating function.
- We applied our approach to rederive known results on binary increasing trees and to obtain an upper bound on recursive trees $O(n/\ln n)$. We have reasons to believe that this bound is already sharp.

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- But the complexity of the recursive generation corresponds to the one of generating integer partitions.

Improved algorithms for the three particular cases

- For some values of the parameters $\phi(z)$ and r, simplifications occur. This is the case for the three models that we have presented.

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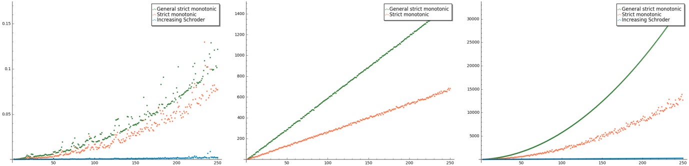

Figure 2: Complexity of uniform samplers for the the three models of increasing Schröder trees. (left) Time complexity of the sampling in milliseconds. (middle) Arithmetic operations on big numbers complexity. (right) Arithmetic operations complexity.

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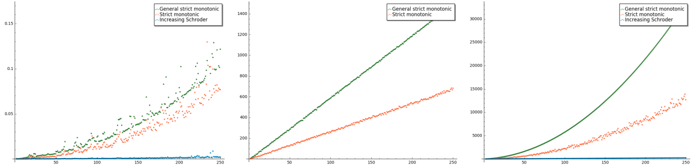

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- For the model of increasing Schröder trees ($\phi(z) = z/(1-z)$ and $r = \{1\}$) we have an incremental process (no recursive generation or operations on big numbers).

Simulation of some parameters

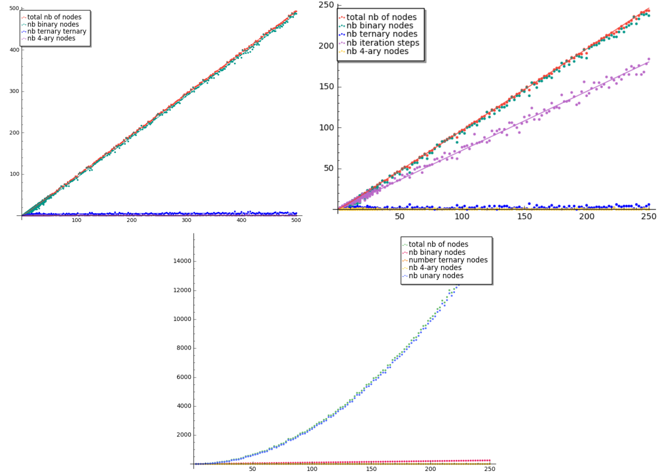

Figure 3: The number of d-ary nodes and the number of iteration steps. (up left) Increasing Schröder trees. (up right) Strict monotonic Schröder trees. (down) general monotonic Schröder trees.